



FHSST Authors

**The Free High School Science Texts:
Textbooks for High School Students
Studying the Sciences
Mathematics
Grades 10 - 12**

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Part I

Basics

Chapter 1

Introduction to Book

1.1 The Language of Mathematics

The purpose of any language, like English or Zulu, is to make it possible for people to communicate. All languages have an alphabet, which is a group of letters that are used to make up words. There are also rules of grammar which explain how words are supposed to be used to build up sentences. This is needed because when a sentence is written, the person reading the sentence understands exactly what the writer is trying to explain. Punctuation marks (like a full stop or a comma) are used to further clarify what is written.

Mathematics is a language, specifically it is the language of Science. Like any language, mathematics has letters (known as numbers) that are used to make up words (known as expressions), and sentences (known as equations). The punctuation marks of mathematics are the different signs and symbols that are used, for example, the plus sign (+), the minus sign (-), the multiplication sign (\times), the equals sign (=) and so on. There are also rules that explain how the numbers should be used together with the signs to make up equations that express some meaning.

Appendix A

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