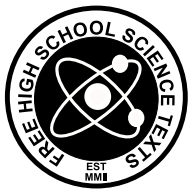

FHSST Authors

The Free High School Science Texts: Textbooks for High School Students Studying the Sciences Mathematics Grades 10 - 12

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Chapter 40

Differential Calculus - Grade 12

40.1 Why do I have to learn this stuff?

Calculus is one of the central branches of mathematics and was developed from algebra and geometry. Calculus is built on the concept of limits, which will be discussed in this chapter. Calculus consists of two complementary ideas: differential calculus and integral calculus. Only differential calculus will be studied. Differential calculus is concerned with the instantaneous rate of change of quantities with respect to other quantities, or more precisely, the local behaviour of functions. This can be illustrated by the slope of a function's graph. Examples of typical differential calculus problems include: finding the acceleration and velocity of a free-falling body at a particular moment and finding the optimal number of units a company should produce to maximize its profit.

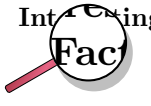
Calculus is fundamentally different from the mathematics that you have studied previously. Calculus is more dynamic and less static. It is concerned with change and motion. It deals with quantities that approach other quantities. For that reason it may be useful to have an overview of the subject before beginning its intensive study.

Calculus is a tool to understand many natural phenomena like how the wind blows, how water flows, how light travels, how sound travels and how the planets move. However, other human activities such as economics are also made easier with calculus.

In this section we give a glimpse of some of the main ideas of calculus by showing how limits arise when we attempt to solve a variety of problems.

Extension: Integral Calculus

Integral calculus is concerned with the accumulation of quantities, such as areas under a curve, linear distance traveled, or volume displaced. Differential and integral calculus act inversely to each other. Examples of typical integral calculus problems include finding areas and volumes, finding the amount of water pumped by a pump with a set power input but varying conditions of pumping losses and pressure and finding the amount of rain that fell in a certain area if the rain fell at a specific rate.



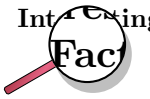
Both Isaac Newton (4 January 1643 – 31 March 1727) and Gottfried Liebnitz (1 July 1646 – 14 November 1716 (Hanover, Germany)) are credited with the 'invention' of calculus. Newton was the first to apply calculus to general physics, while Liebnitz developed most of the notation that is still in use today.

When Newton and Leibniz first published their results, there was some controversy over whether Leibniz's work was independent of Newton's. While Newton derived his results years before Leibniz, it was only some time after Leibniz published in 1684 that Newton published. Later, Newton would claim that Leibniz

got the idea from Newton's notes on the subject; however examination of the papers of Leibniz and Newton show they arrived at their results independently, with Leibniz starting first with integration and Newton with differentiation. This controversy between Leibniz and Newton divided English-speaking mathematicians from those in Europe for many years, which slowed the development of mathematical analysis. Today, both Newton and Leibniz are given credit for independently developing calculus. It is Leibniz, however, who is credited with giving the new discipline the name it is known by today: "calculus". Newton's name for it was "the science of fluxions".

40.2 Limits

40.2.1 A Tale of Achilles and the Tortoise



Zeno (circa 490 BC - circa 430 BC) was a pre-Socratic Greek philosopher of southern Italy who is famous for his paradoxes.

One of Zeno's paradoxes can be summarised by:

> Achilles and a tortoise agree to a race, but the tortoise is unhappy because Achilles is very fast. So, the tortoise asks Achilles for a head-start. Achilles agrees to give the tortoise a 1 000 m head start. Does Achilles overtake the tortoise?

We know how to solve this problem. We start by writing:

$$x_A = v_A t \tag{40.1}$$
$$x_t = 1000\,\mathrm{m} + \mathrm{v_t t} \tag{40.2}$$

where

x_A	distance covered by Achilles
v_A	Achilles' speed
t	time taken by Achilles to overtake tortoise
x_t	distance covered by the tortoise
v_t	the tortoise's speed

If we assume that Achilles runs at 2 m·s^{-1} and the tortoise runs at 0,25 m·s^{-1} then Achilles will overtake the tortoise when both of them have covered the same distance. This means that

Achilles overtakes the tortoise at a time calculated as:

$$\begin{aligned} x_A &= x_t && (40.3)\\ v_A t &= 1000 + v_t t && (40.4)\\ (2\,\mathrm{m}\cdot\mathrm{s}^{-1})t &= 1000\,\mathrm{m} + (0{,}25\,\mathrm{m}\cdot\mathrm{s}^{-1})\mathrm{t} && (40.5)\\ (2\,\mathrm{m}\cdot\mathrm{s}^{-1} - 0{,}25\,\mathrm{m}\cdot\mathrm{s}^{-1})t &= 1000\,\mathrm{m} && (40.6)\\ t &= \frac{1000\,\mathrm{m}}{1\frac{3}{4}\,\mathrm{m}\cdot\mathrm{s}^{-1}} && (40.7)\\ &= \frac{1000\,\mathrm{m}}{\frac{7}{4}\,\mathrm{m}\cdot\mathrm{s}^{-1}} && (40.8)\\ &= \frac{(4)(1000)}{7}\,\mathrm{s} && (40.9)\\ &= \frac{4000}{7}\,\mathrm{s} && (40.10)\\ &= 571\frac{3}{7}\,\mathrm{s} && (40.11) \end{aligned}$$

However, Zeno (the Greek philosopher who thought up this problem) looked at it as follows: Achilles takes

$$t = \frac{1000}{2} = 500\,\mathrm{s}$$

to travel the 1 000 m head start that the tortoise had. However, in this 500 s, the tortoise has travelled a further

$$x = (500)(0{,}25) = 125\,\mathrm{m}.$$

Achilles then takes another

$$t = \frac{125}{2} = 62{,}5\,\mathrm{s}$$

to travel the 125 m. In this 62,5 s, the tortoise travels a further

$$x = (62{,}5)(0{,}25) = 15{,}625\,\mathrm{m}.$$

Zeno saw that Achilles would always get closer but wouldn't actually overtake the tortoise.

40.2.2 Sequences, Series and Functions

So what does Zeno, Achilles and the tortoise have to do with calculus?

Well, in Grades 10 and 11 you studied sequences. For the sequence

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$$

which is defined by the expression

$$a_n = 1 - \frac{1}{n}$$

the terms get closer to 1 as n gets larger. Similarly, for the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

which is defined by the expression

$$a_n = \frac{1}{n}$$

the terms get closer to 0 as n gets larger. We have also seen that the infinite geometric series has a finite total. The infinite geometric series is

$$S_\infty = \sum_{i=1}^{\infty} a_1 . r^{i-1} = \frac{a_1}{1-r} \qquad \text{for} \qquad -1$$

where a_1 is the first term of the series and r is the common ratio.

We see that there are some functions where the value of the function gets close to or **approaches** a certain value.

Similarly, for the function:

$$y = \frac{x^2 + 4x - 12}{x + 6}$$

The numerator of the function can be factorised as:

$$y = \frac{(x + 6)(x - 2)}{x + 6}.$$

Then we can cancel the $x - 6$ from numerator and denominator and we are left with:

$$y = x - 2.$$

However, we are only able to cancel the $x + 6$ term if $x \neq -6$. If $x = -6$, then the denominator becomes 0 and the function is not defined. This means that the domain of the function does not include $x = -6$. But we can examine what happens to the values for y as x gets close to -6. These values are listed in Table 40.1 which shows that as x gets closer to -6, y gets close to 8.

Table 40.1: Values for the function $y = \frac{(x+6)(x-2)}{x+6}$ as x gets close to -6.

x	$y = \frac{(x+6)(x-2)}{x+6}$
-9	-11
-8	-10
-7	-9
-6.5	-8.5
-6.4	-8.4
-6.3	-8.3
-6.2	-8.2
-6.1	-8.1
-6.09	-8.09
-6.08	-8.08
-6.01	-8.01
-5.9	-7.9
-5.8	-7.8
-5.7	-7.7
-5.6	-7.6
-5.5	-7.5
-5	-7
-4	-6
-3	-5

The graph of this function is shown in Figure 40.1. The graph is a straight line with slope 1 and intercept -2, but with a missing section at $x = -6$.

Extension: Continuity

We say that a function is continuous if there are no values of the independent variable for which the function is undefined.

40.2.3 Limits

We can now introduce a new notation. For the function $y = \frac{(x + 6)(x - 2)}{x + 6}$, we can write:

$$\lim_{x \to -6} \frac{(x + 6)(x - 2)}{x + 6} = -8.$$

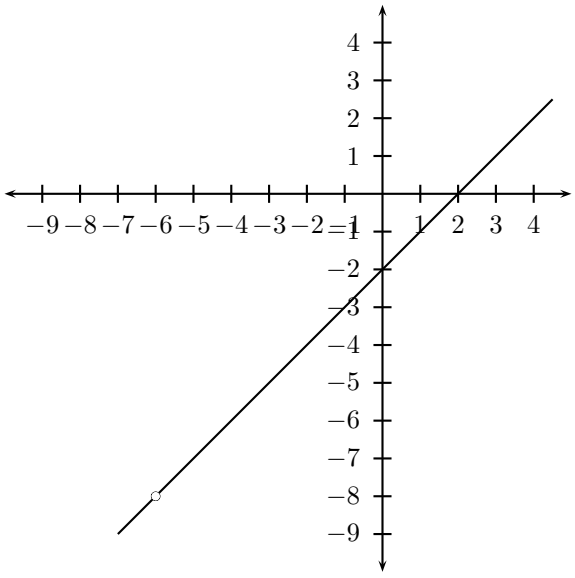


Figure 40.1: Graph of $y = \frac{(x+6)(x-2)}{x+6}$.

This is read: *the limit of* $\frac{(x+6)(x-2)}{x+6}$ *as* x *tends to -6 is 8.*

Activity :: Investigation : Limits
If $f(x) = x + 1$, determine:

f(-0.1)	
f(-0.05)	
f(-0.04)	
f(-0.03)	
f(-0.02)	
f(-0.01)	
f(0.00)	
f(0.01)	
f(0.02)	
f(0.03)	
f(0.04)	
f(0.05)	
f(0.1)	

What do you notice about the value of $f(x)$ as x gets close to 0.

Worked Example 172: Limits Notation

Question: Summarise the following situation by using limit notation: As x gets close to 1, the value of the function

$$y = x + 2$$

gets close to 3.

Answer
This is written as:

$$\lim_{x \to 1} x + 2 = 3$$

in limit notation.

We can also have the situation where a function has a different value depending on whether x approaches from the left or the right. An example of this is shown in Figure 40.2.

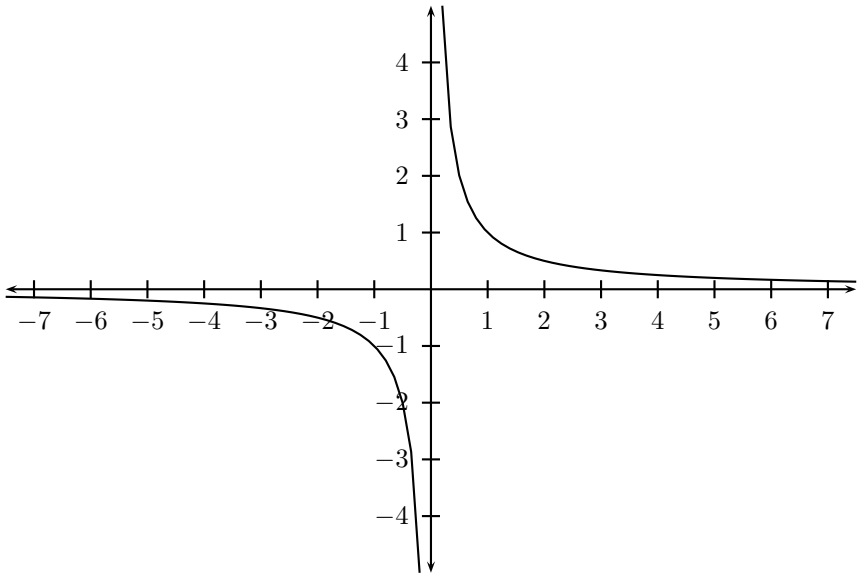


Figure 40.2: Graph of $y = \frac{1}{x}$.

As $x \to 0$ from the left, $y = \frac{1}{x}$ approaches $-\infty$. As $x \to 0$ from the right, $y = \frac{1}{x}$ approaches $+\infty$. This is written in limits notation as:

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

for x approaching zero from the left and

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

for x approaching zero from the right. You can calculate the limit of many different functions using a set method.

Method:

Limits If you are required to calculate a limit like $\lim_{x \to a}$ then:

1. Simplify the expression completely.
2. If it is possible, cancel all common terms.
3. Let x approach the a.

Worked Example 173: Limits

Question: Determine

$$\lim_{x \to 1} 10$$

Answer
Step 1 : Simplify the expression
There is nothing to simplify.
Step 2 : Cancel all common terms
There are no terms to cancel.
Step 3 : Let $x \to 1$ and write final answer

$$\lim_{x \to 1} 10 = 10$$

Worked Example 174: Limits

Question: Determine

$$\lim_{x \to 2} x$$

Answer
Step 1 : Simplify the expression
There is nothing to simplify.
Step 2 : Cancel all common terms
There are no terms to cancel.
Step 3 : Let $x \to 2$ and write final answer

$$\lim_{x \to 2} x = 2$$

Worked Example 175: Limits

Question: Determine

$$\lim_{x \to 10} \frac{x^2 - 100}{x - 10}$$

Answer
Step 1 : Simplify the expression
The numerator can be factorised.

$$\frac{x^2 - 100}{x - 10} = \frac{(x + 10)(x - 10)}{x - 10}$$

Step 2 : Cancel all common terms
$x - 10$ can be cancelled from the numerator and denominator.

$$\frac{(x + 10)(x - 10)}{x - 10} = x + 10$$

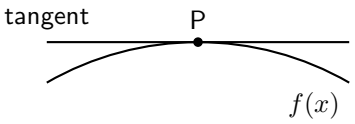
Step 3 : Let $x \to 1$ and write final answer

$$\lim_{x \to 10} \frac{x^2 - 100}{x - 10} = 20$$

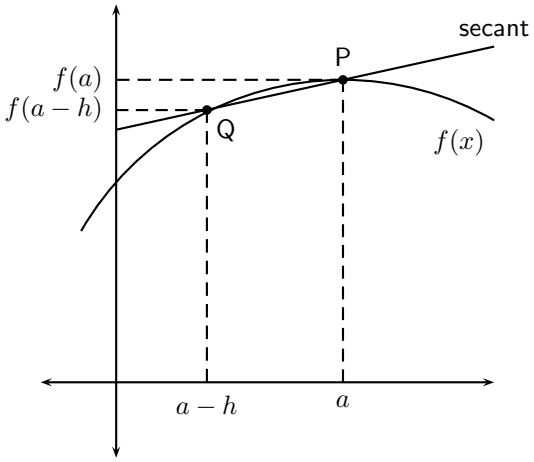
40.2.4 Average Gradient and Gradient at a Point

In Grade 10 you learnt about average gradients on a curve. The average gradient between any two points on a curve is given by the gradient of the straight line that passes through both points. In Grade 11 you were introduced to the idea of a gradient at a single point on a curve. We saw that this was the gradient of the tangent to the curve at the given point, but we did not learn how to determine the gradient of the tangent.

Now let us consider the problem of trying to find the gradient of a tangent t to a curve with equation $y = f(x)$ at a given point P.



We know how to calculate the average gradient between two points on a curve, but we need two points. The problem now is that we only have one point, namely P. To get around the problem we first consider a secant to the curve that passes through point P and another point on the curve Q. We can now find the average gradient of the curve between points P and Q.



If the x-coordinate of P is a, then the y-coordinate is $f(a)$. Similarly, if the x-coordinate of Q is $a - h$, then the y-coordinate is $f(a - h)$. If we choose a as x_2 and $a - h$ as x_1, then:

$$y_1 = f(a - h)$$

$$y_2 = f(a).$$

We can now calculate the average gradient as:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a) - f(a - h)}{a - (a - h)} \tag{40.12}$$

$$= \frac{f(a) - f(a - h)}{h} \tag{40.13}$$

Now imagine that Q moves along the curve toward P. The secant line approaches the tangent line as its limiting position. This means that the average gradient of the secant *approaches* the gradient of the tangent to the curve at P. In (40.13) we see that as point Q approaches point P, h gets closer to 0. When $h = 0$, points P and Q are equal. We can now use our knowledge of limits to write this as:

$$\text{gradient at } P = \lim_{h \to 0} \frac{f(a) - f(a - h)}{h}. \tag{40.14}$$

and we say that the gradient at point P is the limit of the average gradient as Q approaches P along the curve.

Activity :: Investigation : Limits
The gradient at a point x on a curve defined by $f(x)$ can also be written as:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{40.15}$$

Show that this is equivalent to (40.14).

Worked Example 176: Limits

Question: For the function $f(x) = 2x^2 - 5x$, determine the gradient of the tangent to the curve at the point $x = 2$.
Answer
Step 1 : Calculating the gradient at a point
We know that the gradient at a point x is given by:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

In our case $x = 2$. It is simpler to substitute $x = 2$ at the end of the calculation.
Step 2 : Write $f(x+h)$ and simplify

$$\begin{aligned} f(x+h) &= 2(x+h)^2 - 5(x+h) \\ &= 2(x^2 + 2xh + h^2) - 5x - 5h \\ &= 2x^2 + 4xh + 2h^2 - 5x - 5h \end{aligned}$$

Step 3 : Calculate limit

$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \frac{2x^2 + 4xh + 2h^2 - 5x - 5h - (2x^2 - 5x)}{h} \\ &= \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 5x - 5h - 2x^2 + 5x}{h} \\ &= \lim_{h \to 0} \frac{4xh + 2h^2 - 5h}{h} \\ &= \lim_{h \to 0} \frac{h(4x + 2h - 5)}{h} \\ &= \lim_{h \to 0} 4x + 2h - 5 \\ &= 4x - 5 \end{aligned}$$

Step 4 : Calculate gradient at $x = 2$

$$4x - 5 = 4(2) - 5 = 3$$

Step 5 : Write the final answer
The gradient of the tangent to the curve $f(x) = 2x^2 - 5x$ at $x = 2$ is 3.

Worked Example 177: Limits

Question: For the function $f(x) = 5x^2 - 4x + 1$, determine the gradient of the tangent to curve at the point $x = a$.

Answer

Step 1 : Calculating the gradient at a point

We know that the gradient at a point x is given by:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

In our case $x = a$. It is simpler to substitute $x = a$ at the end of the calculation.

Step 2 : Write $f(x+h)$ and simplify

$$\begin{aligned} f(x+h) &= 5(x+h)^2 - 4(x+h) + 1 \\ &= 5(x^2 + 2xh + h^2) - 4x - 4h + 1 \\ &= 5x^2 + 10xh + 5h^2 - 4x - 4h + 1 \end{aligned}$$

Step 3 : Calculate limit

$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \frac{5x^2 + 10xh + 5h^2 - 4x - 4h + 1 - (5x^2 - 4x + 1)}{h} \\ &= \lim_{h \to 0} \frac{5x^2 + 10xh + 5h^2 - 4x - 4h + 1 - 5x^2 + 4x - 1}{h} \\ &= \lim_{h \to 0} \frac{10xh + 5h^2 - 4h}{h} \\ &= \lim_{h \to 0} \frac{h(10x + 5h - 4)}{h} \\ &= \lim_{h \to 0} 10x + 5h - 4 \\ &= 10x - 4 \end{aligned}$$

Step 4 : Calculate gradient at $x = a$

$$10x - 4 = 10a - 5$$

Step 5 : Write the final answer

The gradient of the tangent to the curve $f(x) = 5x^2 - 4x + 1$ at $x = 1$ is $10a - 5$.

Exercise: Limits

Determine the following

1. $$\lim_{x \to 3} \frac{x^2 - 9}{x + 3}$$
2. $$\lim_{x \to 3} \frac{x + 3}{x^2 + 3x}$$
3. $$\lim_{x \to 2} \frac{3x^2 - 4x}{3 - x}$$

4.
$$\lim_{x\to 4} \frac{x^2 - x - 12}{x - 4}$$

5.
$$\lim_{x\to 2} 3x + \frac{1}{3x}$$

40.3 Differentiation from First Principles

The tangent problem has given rise to the branch of calculus called **differential calculus** and the equation:

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

defines the **derivative of the function** $f(x)$**.** Using (40.15) to calculate the derivative is called **finding the derivative from first principles**.

Definition: Derivative
The derivative of a function $f(x)$ is written as $f'(x)$ and is defined by:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{40.16}$$

There are a few different notations used to refer to derivatives. If we use the traditional notation $y = f(x)$ to indicate that the dependent variable is y and the independent variable is x, then some common alternative notations for the derivative are as follows:

$$f'(x) \;=\; y' \;=\; \frac{dy}{dx} \;=\; \frac{df}{dx} \;=\; \frac{d}{dx}f(x) \;=\; Df(x) \;=\; D_x f(x)$$

The symbols D and $\frac{d}{dx}$ are called **differential operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative. It is very important that you learn to identify these different ways of denoting the derivative, and that you are consistent in your usage of them when answering questions.

Important: Though we choose to use a fractional form of representation, $\frac{dy}{dx}$ is a limit and **is not** a fraction, i.e. $\frac{dy}{dx}$ does not mean $dy \div dx$. $\frac{dy}{dx}$ means y differentiated with respect to x. Thus, $\frac{dp}{dx}$ means p differentiated with respect to x. The '$\frac{d}{dx}$' is the "operator", operating on some function of x.

Worked Example 178: Derivatives - First Principles

Question: Calculate the derivative of $g(x) = x - 1$ from first principles.
Answer
Step 1 : Calculating the gradient at a point
We know that the gradient at a point x is given by:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

Step 2 : Write $g(x+h)$ **and simplify**

$$g(x+h) = x+h-1$$

Step 3 : Calculate limit

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{x+h-1-(x-1)}{h} \\ &= \lim_{h\to 0} \frac{x+h-1-x+1}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= \lim_{h\to 0} 1 \\ &= 1 \end{aligned}$$

Step 4 : Write the final answer
The derivative $g'(x)$ of $g(x) = x - 1$ is 1.

Worked Example 179: Derivatives - First Principles

Question: Calculate the derivative of $h(x) = x^2 - 1$ from first principles.
Answer
Step 1 : Calculating the gradient at a point
We know that the gradient at a point x is given by:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

Step 2 : Write $g(x+h)$ and simplify

$$g(x+h) = x+h-1$$

Step 3 : Calculate limit

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{x+h-1-(x-1)}{h} \\ &= \lim_{h\to 0} \frac{x+h-1-x+1}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= \lim_{h\to 0} 1 \\ &= 1 \end{aligned}$$

Step 4 : Write the final answer
The derivative $g'(x)$ of $g(x) = x - 1$ is 1.

Exercise: Derivatives

1. Given $g(x) = -x^2$
   A determine $\frac{g(x+h) - g(x)}{h}$
   B hence, determine
   $$\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$
   C explain the meaning of your answer in (b).
2. Find the derivative of $f(x) = -2x^2 + 3x$ using first principles.
3. Determine the derivative of $f(x) = \frac{1}{x-2}$ using first principles.
4. Determine $f'(3)$ from first principles if $f(x) = -5x^2$.
5. If $h(x) = 4x^2 - 4x$, determine $h'(x)$ using first principles.

40.4 Rules of Differentiation

Calculating the derivative of a function from first principles is very long, and it is easy to make mistakes. Fortunately, there are rules which make calculating the derivative simple.

Activity :: Investigation : Rules of Differentiation
From first principles, determine the derivatives of the following:

1. $f(x) = b$
2. $f(x) = x$
3. $f(x) = x^2$
4. $f(x) = x^3$
5. $f(x) = 1/x$

You should have found the following:

$f(x)$	$f'(x)$
b	0
x	1
x^2	$2x$
x^3	$3x^2$
$1/x = x^{-1}$	$-x^{-2}$

If we examine these results we see that there is a pattern, which can be summarised by:

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \tag{40.17}$$

There are two other rules which make differentiation simpler. For any two functions $f(x)$ and $g(x)$:

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x) \tag{40.18}$$

This means that we differentiate each term separately.

The final rule applies to a function $f(x)$ that is multiplied by a constant k.

$$\frac{d}{dx}[k.f(x)] = kf'(x) \tag{40.19}$$

Worked Example 180: Rules of Differentiation

Question: Determine the derivative of $x - 1$ using the rules of differentiation.
Answer
Step 1 : Identify the rules that will be needed
We will apply two rules of differentiation:

$$\frac{d}{dx}\ (x^n) = nx^{n-1}$$

and

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

Step 2 : Determine the derivative
In our case $f(x) = x$ and $g(x) = 1$.

$$f'(x) = 1$$

and

$$g'(x) = 0$$

Step 3 : Write the final answer
The derivative of $x - 1$ is 1 which is the same result as was obtained earlier, from first principles.

40.4.1 Summary of Differentiation Rules

$$\tfrac{d}{dx}b = 0$$

$$\tfrac{d}{dx}(x^n) = nx^{n-1}$$

$$\tfrac{d}{dx}(kf) = k\tfrac{df}{dx}$$

$$\tfrac{d}{dx}(f + g) = \tfrac{df}{dx} + \tfrac{dg}{dx}$$

Exercise: Rules of Differentiation

1. Find $f'(x)$ if $f(x) = \dfrac{x^2 - 5x + 6}{x - 2}$.
2. Find $f'(y)$ if $f(y) = \sqrt{y}$.
3. Find $f'(z)$ if $f(z) = (z - 1)(z + 1)$.
4. Determine $\frac{dy}{dx}$ if $y = \dfrac{x^3 + 2\sqrt{x} - 3}{x}$.

5. Determine the derivative of $y = \sqrt{x^3} + \frac{1}{3x^3}$.

40.5 Applying Differentiation to Draw Graphs

Thus far we have learnt about how to differentiate various functions, but I am sure that you are beginning to ask, *What is the point of learning about derivatives?* Well, we know one important fact about a derivative: it is a gradient. So, any problems involving the calculations of gradients or rates of change can use derivatives. One simple application is to draw graphs of functions by firstly determine the gradients of straight lines and secondly to determine the turning points of the graph.

40.5.1 Finding Equations of Tangents to Curves

In section 40.2.4 we saw that finding the gradient of a tangent to a curve is the same as finding the slope of the same curve at the point of the tangent. We also saw that the gradient of a function at a point is just its derivative.

Since we have the gradient of the tangent and the point on the curve through which the tangent passes, we can find the equation of the tangent.

Worked Example 181: Finding the Equation of a Tangent to a Curve

Question: Find the equation of the tangent to the curve $y = x^2$ at the point (1,1) and draw both functions.

Answer

Step 1 : Determine what is required

We are required to determine the equation of the tangent to the curve defined by $y = x^2$ at the point (1,1). The tangent is a straight line and we can find the equation by using derivatives to find the gradient of the straight line. Then we will have the gradient and one point on the line, so we can find the equation using:

$$y - y_1 = m(x - x_1)$$

from grade 11 Coordinate Geometry.

Step 2 : Differentiate the function

Using our rules of differentiation we get:

$$y' = 2x$$

Step 3 : Find the gradient at the point (1,1)

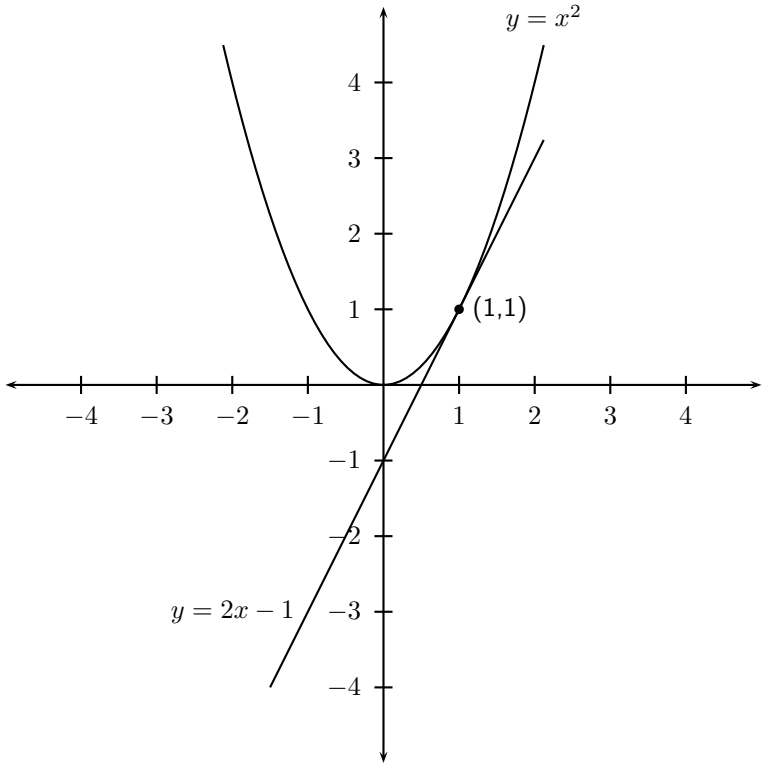
In order to determine the gradient at the point (1,1), we substitute the x-value into the equation for the derivative. So, y' at $x = 1$ is:

$$2(1) = 2$$

Step 4 : Find equation of tangent

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= (2)(x - 1) \\ y &= 2x - 2 + 1 \\ y &= 2x - 1 \end{aligned}$$

Step 5 : Write the final answer
The equation of the tangent to the curve defined by $y = x^2$ at the point (1,1) is $y = 2x - 1$.
Step 6 : Sketch both functions



40.5.2 Curve Sketching

Differentiation can be used to sketch the graphs of functions, by helping determine the turning points. We know that if a graph is increasing on an interval and reaches a turning point, then the graph will start decreasing after the turning point. The turning point is also known as a stationary point because the gradient at a turning point is 0. We can then use this information to calculate turning points, by calculating the points at which the derivative of a function is 0.

Important: If $x = a$ is a turning point of $f(x)$, then:

$$f'(a) = 0$$

This means that the derivative is 0 at a turning point.

Take the graph of $y = x^2$ as an example. We know that the graph of this function has a turning point at (0,0), but we can use the derivative of the function:

$$y' = 2x$$

and set it equal to 0 to find the x-value for which the graph has a turning point.

$$\begin{aligned} 2x &= 0 \\ x &= 0 \end{aligned}$$

We then substitute this into the equation of the graph (i.e. $y = x^2$) to determine the y-coordinate of the turning point:

$$f(0) = (0)^2 = 0$$

This corresponds to the point that we have previously calculated.

Worked Example 182: Calculation of Turning Points

Question: Calculate the turning points of the graph of the function

$$f(x) = 2x^3 - 9x^2 + 12x - 15$$

.

Answer

Step 1 : Determine the derivative of $f(x)$

Using the rules of differentiation we get:

$$f'(x) = 6x^2 - 18x + 12$$

Step 2 : Set $f'(x) = 0$ and calculate x-coordinate of turning point

$$\begin{aligned} 6x^2 - 18x + 12 &= 0 \\ x^2 - 3x + 2 &= 0 \\ (x-2)(x-1) &= 0 \end{aligned}$$

Therefore, the turning points are at $x = 2$ and $x = 1$.

Step 3 : Substitute x-coordinate of turning point into $f(x)$ to determine y-coordinates

$$\begin{aligned} f(2) &= 2(2)^3 - 9(2)^2 + 12(2) - 15 \\ &= 16 - 36 + 24 - 15 \\ &= -11 \end{aligned}$$

$$\begin{aligned} f(1) &= 2(1)^3 - 9(1)^2 + 12(1) - 15 \\ &= 2 - 9 + 12 - 15 \\ &= -10 \end{aligned}$$

Step 4 : Write final answer

The turning points of the graph of $f(x) = 2x^3 - 9x^2 + 12x - 15$ are (2,-11) and (1,-10).

We are now ready to sketch graphs of functions.

Method:

Sketching GraphsSuppose we are given that $f(x) = ax^3 + bx^2 + cx + d$, then there are **five** steps to be followed to sketch the graph of the function:

1. If $a > 0$, then the graph is increasing from left to right, and has a maximum and then a minimum. As x increases, so does $f(x)$. If $a < 0$, then the graph decreasing is from left to right, and has first a minimum and then a maximum. $f(x)$ decreases as x increases.
2. Determine the value of the y-intercept by substituting $x = 0$ into $f(x)$
3. Determine the x-intercepts by factorising $ax^3 + bx^2 + cx + d = 0$ and solving for x. First try to eliminate constant common factors, and to group like terms together so that the expression is expressed as economically as possible. Use the factor theorem if necessary.

4. Find the turning points of the function by working out the derivative $\frac{df}{dx}$ and setting it to zero, and solving for x.

5. Determine the y-coordinates of the turning points by substituting the x values obtained in the previous step, into the expression for $f(x)$.

6. Draw a neat sketch.

Worked Example 183: Sketching Graphs

Question: Draw the graph of $g(x) = x^2 - x + 2$

Answer

Step 1 : Determine the y-intercept

y-intercept is obtained by setting $x = 0$.

$$g(0) = (0)^2 - 0 + 2 = 2$$

Step 2 : Determine the x-intercepts

The x-intercepts are found by setting $g(x) = 0$.

$$\begin{aligned} g(x) &= x^2 - x + 2 \\ 0 &= x^2 - x + 2 \end{aligned}$$

which does not have real roots. Therefore, the graph of $g(x)$ does not have any x-intercepts.

Step 3 : Find the turning points of the function

Work out the derivative $\frac{dg}{dx}$ and set it to zero to for the x coordinate of the turning point.

$$\frac{dg}{dx} = 2x - 1$$

$$\begin{aligned} \frac{dg}{dx} &= 0 \\ 2x - 1 &= 0 \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

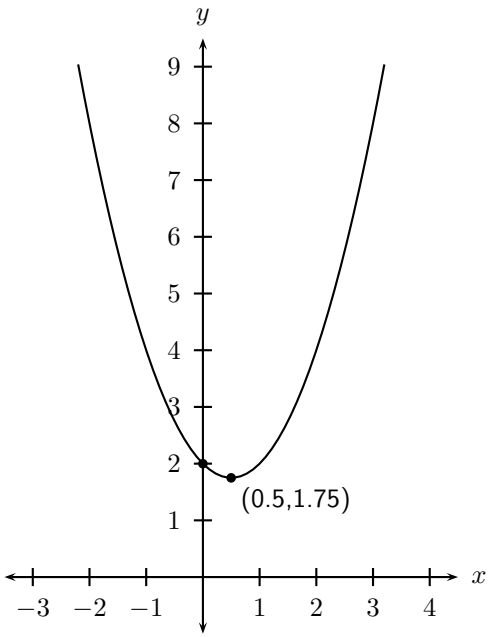
Step 4 : Determine the y-coordinates of the turning points by substituting the x values obtained in the previous step, into the expression for $f(x)$.

y coordinate of turning point is given by calculating $g(\frac{1}{2})$.

$$\begin{aligned} g(\frac{1}{2}) &= (\frac{1}{2})^2 - (\frac{1}{2}) + 2 \\ &= \frac{1}{4} - \frac{1}{2} + 2 \\ &= \frac{7}{4} \end{aligned}$$

The turning point is at $(\frac{1}{2}, \frac{7}{4})$

Step 5 : Draw a neat sketch



Worked Example 184: Sketching Graphs

Question: Sketch the graph of $g(x) = -x^3 + 6x^2 - 9x + 4$.
Answer
Step 1 : Calculate the turning points
Find the turning points by setting $g'(x) = 0$.
If we use the rules of differentiation we get

$$g'(x) = -3x^2 + 12x - 9$$

$$\begin{aligned} g'(x) &= 0 \\ -3x^2 + 12x - 9 &= 0 \\ x^2 - 4x + 3 &= 0 \\ (x-3)(x-1) &= 0 \end{aligned}$$

The x-coordinates of the turning points are: $x = 1$ and $x = 3$.
The y-coordinates of the turning points are calculated as:

$$\begin{aligned} g(x) &= -x^3 + 6x^2 - 9x + 4 \\ g(1) &= -(1)^3 + 6(1)^2 - 9(1) + 4 \\ &= -1 + 6 - 9 + 4 \\ &= 0 \end{aligned}$$

$$\begin{aligned} g(x) &= -x^3 + 6x^2 - 9x + 4 \\ g(3) &= -(3)^3 + 6(3)^2 - 9(3) + 4 \\ &= -27 + 54 - 27 + 4 \\ &= 4 \end{aligned}$$

Therefore the turning points are: $(1,0)$ and $(3,4)$.

Step 2 : Determine the y-intercepts
We find the y-intercepts by finding the value for $g(0)$.

$$\begin{aligned} g(x) &= -x^3 + 6x^2 - 9x + 4 \\ y_{int} = g(0) &= -(0)^3 + 6(0)^2 - 9(0) + 4 \\ &= 4 \end{aligned}$$

Step 3 : Determine the x-intercepts
We find the x-intercepts by finding the points for which the function $g(x) = 0$.

$$g(x) = -x^3 + 6x^2 - 9x + 4$$

Use the factor theorem to confirm that $(x-1)$ is a factor. If $g(1) = 0$, then $(x-1)$ is a factor.

$$\begin{aligned} g(x) &= -x^3 + 6x^2 - 9x + 4 \\ g(1) &= -(1)^3 + 6(1)^2 - 9(1) + 4 \\ &= -1 + 6 - 9 + 4 \\ &= 0 \end{aligned}$$

Therefore, $(x-1)$ is a factor.
If we divide $g(x)$ by $(x-1)$ we are left with:

$$-x^2 + 5x - 4$$

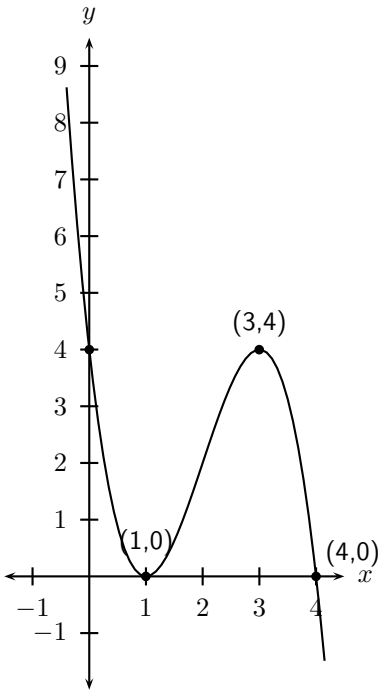
This has factors

$$-(x-4)(x-1)$$

Therefore:

$$g(x) = -(x-1)(x-1)(x-4)$$

The x-intercepts are: $x_{int} = 1, 4$
Step 4 : Draw a neat sketch

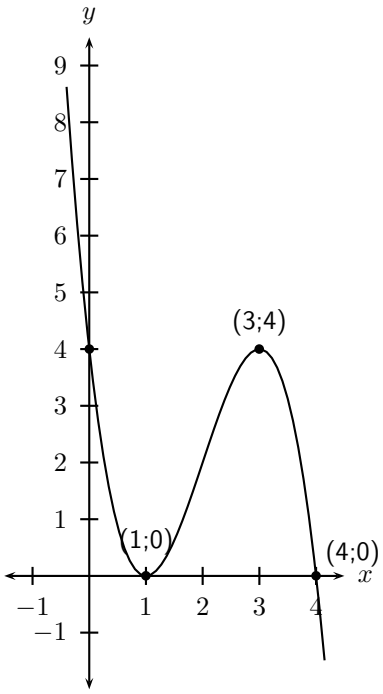


Exercise: Sketching Graphs

1. Given $f(x) = x^3 + x^2 - 5x + 3$:
   A Show that $(x - 1)$ is a factor of $f(x)$ and hence fatorise $f(x)$ fully.
   B Find the coordinates of the intercepts with the axes and the turning points and sketch the graph
2. Sketch the graph of $f(x) = x^3 - 4x^2 - 11x + 30$ showing all the relative turning points and intercepts with the axes.
3. A Sketch the graph of $f(x) = x^3 - 9x^2 + 24x - 20$, showing all intercepts with the axes and turning points.
   B Find the equation of the tangent to $f(x)$ at $x = 4$.

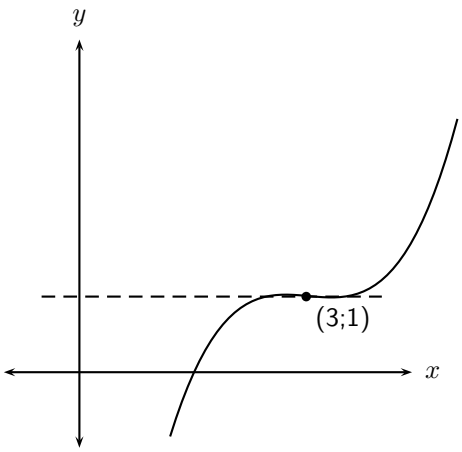
40.5.3 Local minimum, Local maximum and Point of Inflextion

If the derivative ($\frac{dy}{dx}$) is zero at a point, the gradient of the tangent at that point is zero. It means that a turning point occurs as seen in the previous example.



From the drawing the point (1;0) represents a **local minimum** and the point (3;4) the **local maximum**.

A graph has a horizontal **point of inflexion** where the derivative is zero but the sign of the sign of the gradient does not change. That means the graph always increases or always decreases.



From this drawing, the point (3;1) is a horizontal point of inflexion, because the sign of the derivative stays positive.

40.6 Using Differential Calculus to Solve Problems

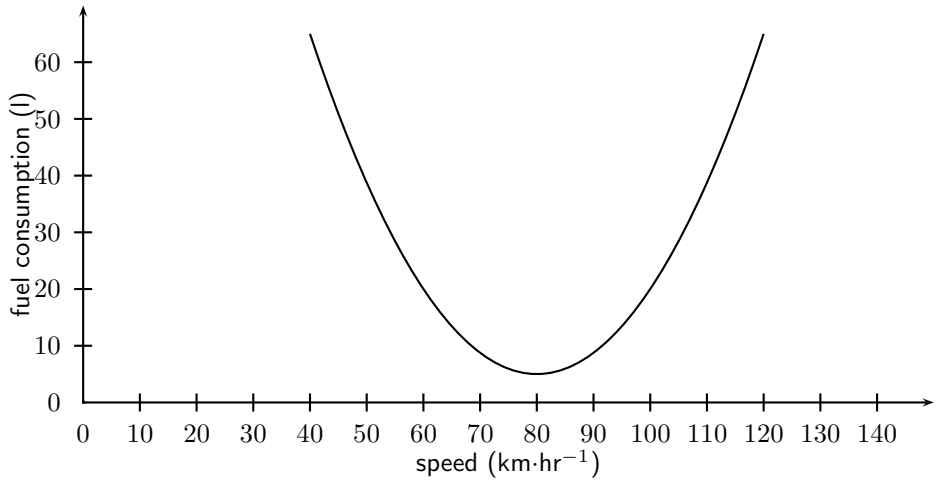
We have seen that differential calculus can be used to determine the stationary points of functions, in order to sketch their graphs. However, determining stationary points also lends itself to the solution of problems that require some variable to be *optimised*.

For example, if fuel used by a car is defined by:

$$f(v) = \frac{3}{80}v^2 - 6v + 245 \tag{40.20}$$

where v is the travelling speed, what is the most economical speed (that means the speed that uses the least fuel)?

If we draw the graph of this function we find that the graph has a minimum. The speed at the minimum would then give the most economical speed.



We have seen that the coordinates of the turning point can be calculated by differentiating the function and finding the x-coordinate (speed in the case of the example) for which the derivative is 0.

Differentiating (40.20), we get:

$$f'(v) = \frac{3}{40}v - 6$$

If we set $f'(v) = 0$ we can calculate the speed that corresponds to the turning point.

$$\begin{aligned} f'(v) &= \frac{3}{40}v - 6 \\ 0 &= \frac{3}{40}v - 6 \\ v &= \frac{6 \times 40}{3} \\ &= 80 \end{aligned}$$

This means that the most economical speed is 80 km·hr^{-1}.

Worked Example 185: Optimisation Problems

Question: The sum of two positive numbers is 10. One of the numbers is multiplied by the square of the other. If each number is greater than 0, find the numbers that make this product a maximum.

Answer

Step 1 : Examine the problem and formulate the equations that are required

Let the two numbers be a and b. Then we have:

$$a + b = 10 \tag{40.21}$$

We are required to minimise the product of a and b. Call the product P. Then:

$$P = a \cdot b \tag{40.22}$$

We can solve for b from (40.21) to get:

$$b = 10 - a \tag{40.23}$$

Substitute this into (40.22) to write P in terms of a only.

$$P = a(10 - a) = 10a - a^2 \tag{40.24}$$

Step 2 : Differentiate

The derivative of (40.24) is:

$$P'(a) = 10 - 2a$$

Step 3 : Find the stationary point

Set $P'(a) = 0$ to find the value of a which makes P a maximum.

$$\begin{aligned} P'(a) &= 10 - 2a \\ 0 &= 10 - 2a \\ 2a &= 10 \\ a &= \frac{10}{2} \\ a &= 5 \end{aligned}$$

Substitute into (40.27) to solve for the width.

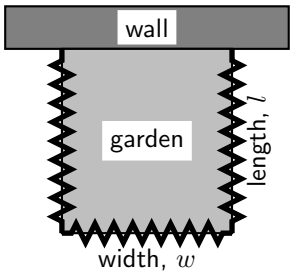
$$\begin{aligned} b &= 10 - a \\ &= 10 - 5 \\ &= 5 \end{aligned}$$

Step 4 : Write the final answer

The product is maximised if a and b are both equal to 5.

Worked Example 186: Optimisation Problems

Question: Michael wants to start a vegetable garden, which he decides to fence off in the shape of a rectangle from the rest of the garden. Michael only has 160 m of fencing, so he decides to use a wall as one border of the vegetable garden. Calculate the width and length of the garden that corresponds to largest possible area that Michael can fence off.



Answer

Step 1 : Examine the problem and formulate the equations that are required
The important pieces of information given are related to the area and modified perimeter of the garden. We know that the area of the garden is:

$$A = w \cdot l \tag{40.25}$$

We are also told that the fence covers only 3 sides and the three sides should add up to 160 m. This can be written as:

$$160 = w + l + l \tag{40.26}$$

However, we can use (40.26) to write w in terms of l:

$$w = 160 - 2l \tag{40.27}$$

Substitute (40.27) into (40.25) to get:

$$A = (160 - 2l)l = 160l - 2l^2 \tag{40.28}$$

Step 2 : Differentiate
Since we are interested in maximising the area, we differentiate (40.28) to get:

$$A'(l) = 160 - 4l$$

Step 3 : Find the stationary point
To find the stationary point, we set $A'(l) = 0$ and solve for the value of l that maximises the area.

$$\begin{aligned} A'(l) &= 160 - 4l \\ 0 &= 160 - 4l \\ \therefore 4l &= 160 \\ l &= \frac{160}{4} \\ l &= 40\,\text{m} \end{aligned}$$

Substitute into (40.27) to solve for the width.

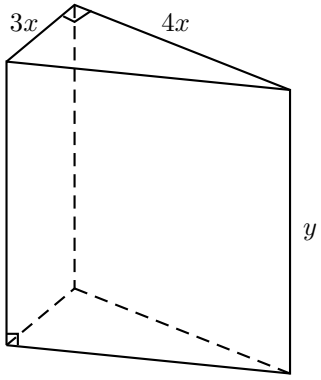
$$\begin{aligned} w &= 160 - 2l \\ &= 160 - 2(40) \\ &= 160 - 80 \\ &= 80\,\text{m} \end{aligned}$$

Step 4 : Write the final answer
A width of 80 m and a length of 40 m will yield the maximal area fenced off.

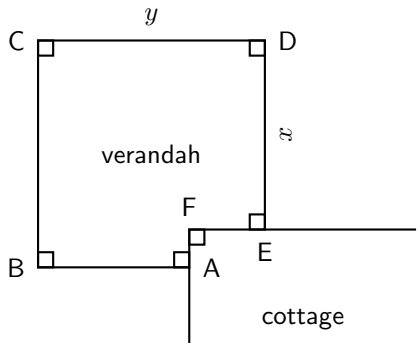
?

Exercise: Solving Optimisation Problems using Differential Calculus

1. The sum of two positive numbers is 20. One of the numbers is multiplied by the square of the other. Find the numbers that make this products a maximum.
2. A wooden block is made as shown in the diagram. The ends are right-angled triangles having sides $3x$, $4x$ and $5x$. The length of the block is y. The total surface area of the block is $3\,600\,\text{cm}^2$.



   A Show that $y = \dfrac{300 - x^2}{x}$.
   B Find the value of x for which the block will have a maximum volume. (Volume = area of base $\times$ height.)
3. The diagram shows the plan for a verandah which is to be built on the corner of a cottage. A railing $ABCDE$ is to be constructed around the four edges of the verandah.



   If $AB = DE = x$ and $BC = CD = y$, and the length of the railing must be 30 metres, find the values of x and y for which the verandah will have a maximum area.

40.6.1 Rate of Change problems

Two concepts were discussed in this chapter: **Average** rate of change $= \frac{f(b)-f(a)}{b-a}$ and **Instantaneous** rate of change $= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. When we mention *rate of change*, the latter is implied. Instantaneous rate of change is the **derivative**. When *Average rate of change* is required, it will be specifically refer to as **average** rate of change.

Velocity is one of the most common forms of rate of change. Again, **average** velocity = **average** rate of change and **instantaneous** velocity = **instantaneous** rate of change = **derivative**. Velocity refers to the increase of distance(s) for a corresponding increade in time (t). The notation commonly used for this is:

$$v(t) = \frac{ds}{dt} = s'(t)$$

Acceleration is the change in velocity for a corersponding increase in time. Therefore, acceleration is the derivative of velocity

$$a(t) = v'(t)$$

This implies that acceleration is the second derivative of the distance(s).

Worked Example 187: Rate of Change

Question: The height (in metres) of a golf ball that is hit into the air after t seconds, is given by $h(t) = 20t = 5t^2$. Determine

1. the average velocity of the ball during the first two seconds
2. the velocity of the ball after 1,5 seconds
3. when the velocity is zero
4. the velocity at which the ball hits the ground
5. the acceleration of the ball

Answer
Step 1 : Average velocity

$$\begin{aligned} Ave\ velocity &= \frac{h(2)-h(0)}{2-0} \\ &= \frac{[20(2)-5(2)^2]-[20(0)-5(0)^2]}{2} \\ &= \frac{40-20}{2} \\ &= 10\ ms^{-1} \end{aligned}$$

Step 2 : Instantaneous Velocity

$$\begin{aligned} v(t) &= \frac{dh}{dt} \\ &= 20-10t \end{aligned}$$

Velocity after 1,5 seconds:

$$\begin{aligned} v(1{,}5) &= 20-10(1{,}5) \\ &= 5\ ms^{-1} \end{aligned}$$

Step 3 : Zero velocity

$$\begin{aligned} v(t) &= 0 \\ 20 - 10t &= 0 \\ 10t &= 20 \\ t &= 2 \end{aligned}$$

Therefore the velocity is zero after 2 seconds
Step 4 : Ground velocity
The ball hits the ground when $h(t) = 0$

$$\begin{aligned} 20t - 5t^2 &= 0 \\ 5t(4 - t) &= 0 \\ t = 0 \quad &or \quad t = 4 \end{aligned}$$

The ball hits the ground after 4 seconds. The velocity after 4 seconds will be:

$$\begin{aligned} v(4) &= h'(4) \\ &= 20 - 10(4) \\ &= 20\ ms^{-1} \end{aligned}$$

The ball hits the gound at a speed of $20ms^{-1}$
Step 5 : Acceleration

$$\begin{aligned} a &= v'(t) \\ &= -10\ ms^{-1} \end{aligned}$$

40.7 End of Chapter Exercises

1. Determine $f'(x)$ from **first principles** if:

$$f(x) = x^2 - 6x$$
$$f(x) = 2x - x^2$$

2. Given: $f(x) = -x^2 + 3x$, find $f'(x)$ using *first principles*.
3. Determine $\frac{dx}{dy}$ if:

   A
   $$y = (2x)^2 - \frac{1}{3x}$$

   B
   $$y = \frac{2\sqrt{x} - 5}{\sqrt{x}}$$

4. Given: $f(x) = x^3 - 3x^2 + 4$

   A Calculate $f(-1)$, and hence solve the equation $f(x) = 0$

   B Determine $f'(x)$

   C Sketch the graph of f neatly and clearly, showing the co-ordinates of the turning points as well as the intercepts on both axes.

D Determine the co-ordinates of the points on the graph of f where the gradient is 9.

5. Given: $f(x) = 2x^3 - 5x^2 - 4x + 3$. The x-intercepts of f are: (-1;0) ($\frac{1}{2}$;0) and (3;0).

   A Determine the co-ordinates of the turning points of f.

   B Draw a neat sketch graph of f. Clearly indicate the co-ordinates of the intercepts with the axes, as well as the co-ordinates of the turning points.

   C For which values of k will the equation $f(x) = k$, have exactly two real roots?

   D Determine the equation of the tangent to the graph of $f(x) = 2x^3 - 5x^2 - 4x + 3$ at the point where $x = 1$.

6. A Sketch the graph of $f(x) = x^3 - 9x^2 + 24x - 20$, showing all intercepts with the axes and turning points.

   B Find the equation of the tangent to $f(x)$ at $x = 4$.

7. Calculate:
$$\lim_{x \to 1} \frac{1 - x^3}{1 - x}$$

8. Given:
$$f(x) = 2x^2 - x$$

   A Use the definition of the derivative to calculate $f'(x)$.

   B Hence, calculate the co-ordinates of the point at which the gradient of the tangent to the graph of f is 7.

9. If $xy - 5 = \sqrt{x^3}$, determine $\frac{dx}{dy}$

10. Given: $g(x) = (x^{-2} + x^2)^2$. Calculate $g'(2)$.

11. Given: $f(x) = 2x - 3$

    A Find: $f^{-1}(x)$

    B Solve: $f^{-1}(x) = 3f'(x)$

12. Find $f'(x)$ for each of the following:

    A $f(x) = \dfrac{\sqrt[5]{x^3}}{3} + 10$

    B $f(x) = \dfrac{(2x^2 - 5)(3x + 2)}{x^2}$

13. Determine the minimum value of the sum of a *positive* number and its reciprocal.

14. If the displacement s (in metres) of a particle at time t (in seconds) is governed by the equation $s = \frac{1}{2}t^3 - 2t$, find its acceleration after 2 seconds. (Acceleration is the rate of change of velocity, and velocity is the rate of change of displacement.)

15. A After doing some research, a transport company has determined that the rate at which petrol is consumed by one of its large carriers, travelling at an average speed of x km per hour, is given by:
$$P(x) = \frac{55}{2x} + \frac{x}{200} \quad \text{litres per kilometre}$$

    i. Assume that the petrol costs R4,00 per litre and the driver earns R18,00 per hour (travelling time). Now deduce that the total cost, C, in Rands, for a 2 000 km trip is given by:
$$C(x) = \frac{256000}{x} + 40x$$

    ii. Hence determine the average speed to be maintained to effect a minimum cost for a 2 000 km trip.

B During an experiment the temperature T (in degrees Celsius), varies with time t (in hours), according to the formula:

$$T(t) = 30 + 4t - \frac{1}{2}t^2 \quad t \in [1; 10]$$

i. Determine an expression for the rate of change of temperature with time.

ii. During which time interval was the temperature dropping?

16. The depth, d, of water in a kettle t minutes after it starts to boil, is given by $d = 86 - \frac{1}{8}t - \frac{1}{4}t^3$, where d is measured in millimetres.

A How many millimetres of water are there in the kettle just before it starts to boil?

B As the water boils, the level in the kettle drops. Find the *rate* at which the water level is decreasing when $t = 2$ minutes.

C How many minutes after the kettle starts boiling will the water level be dropping at a rate of $12\frac{1}{8}$ mm/minute?

Appendix A

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